APPLICATION OF THE INTEGRAL DIFFUSION EQUATIONS TO THE INVESTIGATION OF TURBULENT TRANSPORT

A. T. Onufriev

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 4, pp. 59–65, 1967

In recent years there have appeared several experimental studies [1–5] which have shown that there are cases of turbulent flow with an asymmetric distribution of the flow velocity and in which at the point where the velocity derivative is zero the turbulent shear stress is not zero. This raises the question of the connection of the Reynolds stress tensor with the characteristics of the average flow. The relationships used in the usual mixing length theory connect the shear stress with the local value of the flow velocity derivative and are not consistent with the experimental results mentioned above. These relationships are based on the assumption that the mixing length is small in comparison with the characteristic length of the flow. Experiment shows that this assumption is not justified [6].

Thus, turbulent diffusion refers to the case of diffusion with a large mean free path. In addition to the concept of "gradient diffusion," there is also the concept of "bulk convection" or "integral diffusion" [10], which means a transfer mechanism in which the shear stress is not expressed in terms of the velocity gradient. The generalization of mixing length theory proposed in [11–14] is based on the very simple kinetic equation which was used for the examination of turbulent transfer problems in [8] and which is encountered in the treatment of transport problems in gases, neutron diffusion, and radiative energy transfer.

The proposed generalization of mixing length theory employs an analogy with the indicated processes and permits the derivation of formulas which are valid for large mean free paths. In the case of small mean free paths the obtained relationships lead to the relationships for diffusion in a continuous medium and, in particular, to the relationships of the Prandtl mixing length theory. The integral diffusion model is a phenomenological semiempirical theory in which empirical constants and several hypotheses common in mixing length theory are used. A very general analysis of the expression for the shear stress leads to the conclusion that if the flow is asymmetric over a distance comparable with the "mixing length" the points at which the velocity derivative and the turbulent shear stress are zero do not coincide [12]. Hence, it is to be hoped that the integral diffusion model will allow treatment of the above questions, which cause difficulty in the case of ordinary mixing length theory. Incompressible turbulent flow is considered.

1. Equations. The hydrodynamic fields are decomposed into their mean values and pulsations. The statistical characteristics of the fields are connected by relationships obtained by averaging the hydrodynamic equations for a viscous fluid. For an incompressible fluid in the absence of external forces the continuity and motion equations have the form (see [15], for instance)

$$\partial \rho U_\alpha / \partial x_\alpha = 0\,, \tag{1.1}$$

$$\rho \frac{d}{dt} U_i + \frac{\partial}{\partial x_\alpha} P_{\alpha i} = -\frac{\partial}{\partial x_i}\langle P\rangle - \frac{\partial}{\partial x_\alpha}\langle \overset{\circ}{\sigma}_{i\alpha}\rangle. \tag{1.2}$$

Reynolds' equations contained components of the additional stress tensor. For new unknown values of the Reynolds stresses we can obtain equations, particularly the energy balance equation for the pulsational motion

$$\frac{d}{dt} E + \frac{\partial}{\partial x_\alpha} q_\alpha + P_{\alpha\beta}\frac{\partial}{\partial x_\alpha} U_\beta = \left\langle p \frac{\partial u_\alpha}{\partial x_\alpha}\right\rangle -$$
$$- \frac{\partial}{\partial x_\alpha}(\langle p u_\alpha\rangle - \langle u_\beta \sigma_{\alpha\beta}\rangle) - \left\langle \sigma_{\alpha\beta}\frac{\partial u_\beta}{\partial x_\alpha}\right\rangle,$$
$$E = {}^1/_2\,\rho\,\langle u_\alpha u_\alpha\rangle,\ P_{\alpha\beta} = \rho\,\langle u_\alpha u_\beta\rangle,$$
$$q_\alpha = {}^1/_2\,\rho\,\langle u_\alpha u_k u_k\rangle. \tag{1.3}$$

The symbol $\langle\ \rangle$ denotes averaging, σ_{ij}° is the viscous stress tensor, σ_{ij} is the viscous stress tensor for pulsational components of the velocity, $P_{\alpha\beta}$ is the additional stress tensor, and q_α is the energy flux density of the pulsational motion. The obtained system is open, since the values of the additional stresses are not expressed in terms of averaged characteristics of the hydrodynamic field and the number of unknowns exceeds the number of equations.

If we completely neglect molecular motion we are left with equations characterizing the pulsational turbulent flow

$$\frac{\partial}{\partial x_\alpha}\rho U_\alpha = 0, \quad \rho\frac{d}{dt}U_i + \frac{\partial P_{i\alpha}}{\partial x_\alpha} = 0,$$
$$\frac{d}{dt}E + \frac{\partial}{\partial x_\alpha} q_\alpha + P_{\alpha\beta}\frac{\partial}{\partial x_\alpha}U_\beta = 0. \tag{1.4}$$

A comparison of the system of equations (1.4) describing the pulsational turbulent flow with the system of equations describing the motion of a gas [10] reveals their complete external analogy. The velocity of the turbulent flow pulsations can be compared with the relative velocity of thermal motion of the molecules. The system of equations describing the motion of a gas is closed by establishment of a relationship between the stress tensor and the energy flux density vector and the characteristics of the mean motion after determination of the distribution function. We can suggest that some of the methods developed in the kinetic theory of gases can be used to construct the turbulent stress tensor.

Below we describe an approach to the derivation of relationships for the additional stress tensor and the energy flux density of pulsational motion in the phenomenological theory of turbulent transport. In [11–14] a model of turbulent transport was constructed on the basis of physical arguments and by analogy with radiative energy transfer and neutron transport, making direct use of the macroscopic flow characteristics. The transport model constructed below is based on the use of a distribution function which satisfies a kinetic equation of simple form. This equation has been discussed and used in many problems relating to the flow of a rarefied gas [18–21]. The system of equations of turbulent motion of the liquid (1.1), (1.2), and (1.3), however, is still open.

2. Description of transport model. An idea of the process of turbulent transport can be obtained with the aid of a rough scheme. We assume that at each point in the flow the size of the formations involved in pulsational motion ("moles") is a characteristic quantity and as a characteristic of the rate of turbulent mixing of the "moles" with the surrounding medium we can introduce a quantity analogous to the mean free path — the "mixing length."

We assume that the velocity distribution of the pulsations can be characterized by a distribution function f, satisfying the relaxation equation, which in the steady-state case has the form [20]

$$c_s \partial f / \partial s = A\,(f_0 - f). \qquad (2.1)$$

The first term on the right side characterizes the creation of moles at some point in the flow. The velocity distribution of these moles is assumed to be normal,

$$f_0 = \frac{\rho}{(\pi h)^{3/2}} \exp\left(-\frac{(c_k - U_k)^2}{h}\right).$$

This corresponds to the assumption of local thermodynamic equilibrium in the treatment of radiative energy transfer. The parameter h characterizes the energy of pulsational motion at this point

$$h = {}^2/_3\,\langle C^2\rangle = {}^4/_3\,E / \rho\,,$$

where C is the velocity of true motion in a coordinate system moving with the mean velocity of the flow.

Equation (2.1) can be integrated along the direction s and for an infinite flow we obtain

$$f(M_0) = \int_0^\infty f_0(s)\,\Pi\left(0,\ s;\ \frac{c}{A}\right)\frac{A\,ds}{c}$$

$$\Pi\left(0,\ s;\ \frac{c}{A}\right) = \exp\left[-\tau\left(0,\ s;\ \frac{c}{A}\right)\right],$$

$$\tau\left(0,\ s,\ \frac{c}{A}\right) = \int_0^s \frac{A\,ds}{c}. \qquad (2.2)$$

Here s is the distance along the ray from the point M_0, $f(M_0)$ is the direction function at the point M_0. The distribution function, which characterizes all the turbulent pulsations of velocity in the vicinity of a point in the flow, is not an equilibrium function and will tend to an equilibrium value only when the mixing length tends to zero.

By using expression (2.2) we can obtain formulas for the components of the turbulent stress tensor P_{ij} and the energy flux density of pulsational motion q_i

$$P_{ij}(M_0) = $$

$$= \int [c_i - U_i(M_0)]\,[c_j - U_j(M_0)]\,f(M_0)\,dc_x dc_y dc_z,$$

$$q_i(M_0) = \frac{1}{2}\int [c_i - U_i(M_0)]\times$$

$$\times [c_k - U_k(M_0)]^2\,f(M_0)\,dc_x\,dc_y\,dc_z \qquad (2.3)$$

(summation is performed according to the double indices).

The use of these expressions, which should presumably be valid in the general case, entails awkward calculations.

By introducing a number of simplifications we can obtain the expressions derived previously in the cited works. Firstly, we assume that the difference in the mean velocity at two points in the flow which still have a considerable effect on one another is small in relation to the mean velocity of pulsational motion. This restriction is not always fulfilled with the required strictness. For instance, in the layer of constant shear stress, where a logarithmic law is valid for the profile of the mean velocity, estimates show that this ratio is approximately 0.2 at a distance of one length, and 0.6 at a distance of four lengths. Secondly, the expression for the coefficient characterizing the rate of turbulent mixing of the moles with the surrounding medium is assumed to depend only on the coordinate Λ(s). Qualitatively this hypothesis does not introduce any changes. The quantitative error will presumably lie within the limits of validity of the other hypotheses, particularly since the obtained expressions contain empirical coefficients. The second simplification is similar to the introduction of a mean absorption coefficient in problems of radiative energy transfer. When the absorption coefficient does not vary too greatly the numerical error is smaller. More accuracy can be obtained by introducing division into several intervals in the integration with respect to c. The value of Λ will be determined from additional considerations.

It is worthwhile as a first approximation to use simpler expressions in the calculations. This allows a qualitative assessment of the suitability of the obtained approximation. We can also hope for satisfactory quantitative results in view of the fact that Prandtl's mixing length theory, which entails even more restrictions, gives satisfactory results in the treatment of turbulent transfer in many cases.

After simplifications we obtain the following expression for the additional stress:

$$P_{ij} = \int_{4\pi}\int_0^\infty\left[\int_0^\infty \rho C_i C_j f_0(s)\,C^2\,dC\right]\Pi(0,\ s;\ \Lambda)\,\frac{ds}{\Lambda}\,d\Omega \approx$$

$$\approx \int_{4\pi}\int_0^\infty\left[\int_0^\infty \rho\,(\pi h)^{-3/2} C_i C_j \times\right.$$

$$\left.\times \exp\frac{-C^2}{h}\,C^2\left(1 + \frac{2}{h}\,C_k \Delta U_k\right)dC\right]\times$$

$$\times \Pi(0,\ s;\ \Lambda)\,\frac{ds}{\Lambda}\,d\Omega \quad (\Lambda = c/A)\,, \qquad (2.4)$$

$$q_i = \frac{1}{2}\int_{4\pi}\int_0^\infty\left[\int_0^\infty \rho\,(\pi h)^{-3/2} C_i C^4 \times\right.$$

$$\left.\times \exp\frac{-C^2}{h}\left(1 + \frac{2}{h}\,C_k \Delta U_k\right)dC\right]\Pi(0,\ s;\ \Lambda)\,\frac{ds}{\Lambda}\,d\Omega$$

$$(C_k = c_k - U_k)\,. \qquad (2.5)$$

3. Value of shear stress. From expression (2.4) we obtain (the y-axis is directed upward, angle θ is measured from the positive direction of the y-axis, angle φ is measured from the positive direction of the x-axis in the xz-plane), with $C_k = -C\cos(s, x_k)$

$$P_{ij} = \int_{4\pi}\int_0^\infty \cos(s,\ x_i)\cos(s,\ x_j)\times$$

$$\times\left[\frac{3}{8\pi}\,\rho h - \frac{2}{\pi^{3/2}}\,\rho h^{1/2}\Delta U_k \cos(s,\ x_k)\right]\times$$

$$\times \Pi(0,\ s;\ \Lambda)\,\frac{ds}{\Lambda}\,d\Omega, \qquad d\Omega = \sin\theta\,d\theta\,d\varphi. \qquad (3.1)$$

We consider the limiting case of small mixing length Λ. The quantities h(s) and $\Delta U_k(s) = U_k(s) - U_k(0)$ can be expanded in a series in s and, keeping terms of the order of s, we obtain

$$P_{xx} = -2\frac{\partial U_x}{\partial x}\frac{4}{15}\rho\Lambda\langle C\rangle + \frac{\rho h}{2} \qquad \left(\langle C\rangle = \frac{2}{\pi^{1/2}}h^{1/2}\right)$$

$$P_{xy} = -(\partial U_x/\partial y + \partial U_y/\partial x)\,{}^4/_{15}\,\rho\Lambda\langle C\rangle \text{ etc}, \qquad (3.2)$$

i.e., in the case of a small mixing length we obtain the usual relationships which are valid for small mean free path in a medium of constant density, which leads to the relationships of Prandtl's mixing length theory.

In the case of an arbitrary value of Λ the turbulent shear stress is described by expression (3.1), which we further simplify for the case of plane flow ($\Delta U_y = \Delta U_z = 0$). In this case the expression for P_{xy} will be of interest,

$$P_{xy}(y) = -\int_y^\infty \rho\Delta U\langle C\rangle\{E_2(\tau(y,\zeta;\Lambda)) - E_4(\tau(y,\zeta;\Lambda))\}\frac{d\zeta}{\Lambda} + \int_{-\infty}^{y}\rho\Delta U\langle C\rangle\{E_2(\tau(\zeta,y;\Lambda)) - E_4(\tau(\zeta,y;\Lambda))\}\frac{d\zeta}{\Lambda}. \qquad (3.3)$$

Here E_2 and E_4 are integroexponential functions.

We can obtain an even simpler expression for the turbulent shear stress, which retains all the qualitative features, if we use the mean-cosine approximation ($\langle\cos\theta\rangle = 2/3$)

$$P_{xy}(y) \approx -\frac{1}{4}\int_y^\infty \rho\Delta U\langle C\rangle\Pi(y,\zeta;L)\frac{d\zeta}{L} + \frac{1}{4}\int_{-\infty}^{y}\rho\Delta U\langle C\rangle\Pi(\zeta,y;L)\frac{d\zeta}{L}. \qquad (3.4)$$

For cases in which there is no strong asymmetry in the flow Buleev [11] introduced an "integral diffusion coefficient" approximation. In expression (3.3) the velocity difference ΔU is expanded in a series and then

$$P_{xy} = -\rho\,\varepsilon\,\partial U/\partial y,$$

$$\varepsilon = \frac{1}{4}\int_{-\infty}^{\infty}\langle C\rangle|\zeta - y|\exp\left(-\left|\int_\zeta^y \frac{dz}{L}\right|\right)\frac{d\zeta}{L}. \qquad (3.5)$$

The use of this model gives results which agree satisfactorily with the results of mixing length theory for flows in a boundary layer, a plane channel, and a tube [11, 14], and also enables the treatment of several problems of flow in a tube [11], and a plane turbulent Couette flow [14], which cannot be satisfactorily solved by the usual mixing length theory without a special choice of constants.

If the mixing length is small in comparison with the length in which there is an appreciable change of $\langle C\rangle$, then

$$P_{xy} = -{}^1/_2\,\rho\,\langle C\rangle L\,\partial U/\partial y. \qquad (3.6)$$

4. Energy flux density of pulsational motion. From expression (2.5) for a plane layer we obtain

$$q_y(y) = -\int_y^\infty \frac{\rho h^{3/2}}{\sqrt{\pi}}E_2(\tau(y,\zeta;\Lambda))\frac{d\zeta}{\Lambda} + \int_{-\infty}^{y}\frac{\rho h^{3/2}}{\sqrt{\pi}}E_2(\tau(\zeta,y;\Lambda))\frac{d\zeta}{\Lambda}. \qquad (4.1)$$

In the mean-cosine approximation we obtain the expression

$$q_y(y) \approx -\int_y^\infty \frac{\rho h^{3/2}}{2\sqrt{\pi}}\Pi(y,\zeta;L)\frac{d\zeta}{L} + \int_{-\infty}^{y}\frac{\rho h^{3/2}}{2\sqrt{\pi}}\Pi(\zeta,y;L)\frac{d\zeta}{L} \qquad (4.2)$$

which in the limiting case of small mixing length takes the form

$$q_y \sim -{}^3/_8\,\rho\langle C\rangle L\,\partial h/\partial y. \qquad (4.3)$$

5. Shear stress in asymmetric flow. At the beginning of this paper we referred to studies in which the structure of an asymmetric flow had been investigated. Paper [5], for instance, dealt with the problem of mixing of two parallel streams flowing initially in separate plane channels at different velocities and then flowing together into a channel without a dividing partition. If we consider a plane passing through a point N at which $\partial U/\partial y = 0$, we find that as a result of asymmetry of the distribution of flow velocity and mean-square value of the pulsational velocity relative to this plane the value of the turbulent shear stress is not zero at the point N. The coordinate of the point at which the velocity derivative is zero is shifted systematically relative to the point where the shear stress is zero towards greater intensity of turbulence. This casts doubt on the hypothesis adopted hitherto in the semiempirical mixing length theory, viz., that the additional stress tensor is connected with the flow parameters by an expression of the form of (3.2).

The integral diffusion method enables us to describe this result [12]. If at a distance of several mixing lengths from the plane passing through point N, the distribution of the velocity field and sources of creation of turbulent pulsations are asymmetric, the shear stress will not be zero and the point where $P_{xy} = 0$ will be shifted towards sources of weaker intensity. This fundamental possibility, inherent to the integral diffusion method, provides a basis for its application to the calculation of a flow of the considered type. If a satisfactory agreement between calculation and experiment is obtained, this will be confirmation of the validity of the main hypotheses on which the integral diffusion theory is based.

6. System of equations in boundary-layer approximation. For a boundary-layer flow in which transport is across the direction of flow we can use the relationships obtained for plane flow. Then the system of equations has the form

$$\frac{\partial\rho U_x}{\partial x} + \frac{\partial\rho U_y}{\partial y} = 0,$$

$$\rho U_x\frac{\partial U_x}{\partial x} + \rho U_y\frac{\partial U_x}{\partial y} = -\frac{dp}{dx} - \frac{\partial}{\partial y}(\rho\langle u_x u_y\rangle) + \mu\frac{\partial^2 U_x}{\partial y^2},$$

$$\rho U_x\frac{\partial}{\partial x}\left(\frac{3}{4}h\right) + \rho U_y\frac{\partial}{\partial y}\left(\frac{3}{4}h\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\langle\rho C^2 u_y\rangle + \langle p u_y\rangle\right) = \frac{1}{2}\mu\frac{\partial^2}{\partial y^2}\left(\frac{3}{4}h\right) - \mu\left\langle\left(\frac{\partial u_j}{\partial x_i}\right)^2\right\rangle - \rho\langle u_x u_y\rangle\frac{\partial U_x}{\partial y},$$

$$\rho\langle u_x u_y\rangle = -\frac{4}{3\sqrt{\pi}}\int_y^\infty \rho\Delta U_x(\zeta)\,h^{1/2}\Pi(y,\zeta;L)\frac{d\zeta}{L} +$$

$$+\frac{4}{3\sqrt{\pi}}\int_{-\infty}^{y}\rho\Delta U_x(\zeta)\,h^{1/2}\Pi(\zeta, y; L)\frac{d\zeta}{L},$$

$$\frac{1}{2}\langle\rho C^2 u_y\rangle = -\int_{y}^{\infty}\frac{\rho h^{3/2}}{2\sqrt{\pi}}\Pi(y, \zeta; L)\frac{d\zeta}{L} +$$

$$+\int_{-\infty}^{y}\frac{\rho h^{3/2}}{2\sqrt{\pi}}\Pi(\zeta, y; L)\frac{d\zeta}{L}.$$

As regards the terms $\langle pu_y \rangle$ $\mu\langle(\partial u_j/\partial_{xi})^2\rangle$, and the quantity L we must make certain assumptions. The equations of the energy of pulsational motion were analyzed in [22–27], where either approximate equations were proposed, or dimensionality arguments were used. In particular, the expression proposed in [25] was

$$\mu\,\langle(\partial u_j\,/\,\partial x_i)^2\rangle \sim k_2\mu h\,/\,L^2 + k_3\rho h^{3/2}\,/\,L\,.$$

Here k_2 and k_3 are empirical constants.

Equations [22, 25] or experimental relationships have been proposed for the determination of the mixing length. For developed flow in a channel of constant cross section Obukhov [28] gave relationships for L which are satisfactory in many cases. In connection with the integral diffusion method it is of interest to note that in [28] the treatment involved an internal geometry in which the element was similar to the "optical thickness" element ds/L in the integral diffusion method. Since the probability of interaction does not depend on the geometric coordinate, but on the value of τ (0, s; L), which is different along rays in different directions, then anisotropy appears in the integral diffusion model, as in the internal geometry case.

REFERENCES

1. J. Mathieu, "Contribution a l'étude aérothermique d'un jet plan évoluant en présence d'une paroi," Thes. Sci. Phys., Grenoble, no. 165, p. 1–85, 1959.
2. W. H. Schwarz and W. P. Cosart, "The two-dimensional turbulent wall-jet," J. Fluid Mech., vol. 10, 481–495, 1961.
3. Z. Sakipov, "An experimental investigation of semi-infinite jets," collection: Applied Thermophysics [in Russian], Izd-vo AN Kazakh SSR, pp. 29–46, 1964.
4. M. C. Beguier, "Ecoulements dissymmétriques en régime turbulent," Compt. rend. Hebdomadaires Séances Acad. sci., vol. 260, no. 21, pp. 5460–5463, 1965.
5. M. C. Bequier, "Mesures des tensions de Reynolds dans un écoulement dissymétrique en régime turbulent incompressible," J. Mécan., vol. 4, no. 3, p. 319–334, 1965.
6. G. K. Batchelor, "Note on free turbulent flows with special reference to the two-dimensional wake," J. Aeronaut. Sci., vol. 17, no. 3, pp. 441–445, 1950.
7. A. A. Townsend, The Structure of Turbulent Shear Flow [Russian translation], Izd. inostr. lit., 1959.
8. G. B. Schubauer and C. M. Tchen, "Turbulent flow," in: Turbulent Flows and Heat Transfer, ed. C. C. Lin, Izd. inostr. lit., chapter 2, pp. 83–205, 1963.
9. J. O. Hinze, Turbulence—An Introduction to its Mechanism and Theory [Russian translation], Fizmatgiz, 1963.
10. J. C. Schönfeld, "Integral diffusivity," J. Geophys. Res., vol. 67, no. 8, pp. 3187–3199, 1962.
11. N. I. Buleev, "A theoretical model of the mechanism of turbulent transport in liquid flows," collection: Heat Transfer [in Russian], Izd-vo AN SSSR, pp. 64–98, 1962.
12. A. T. Onufriev, "A model of nonequilibrium processes in some problems of the mechanics of continuous media," PMTF, no. 1, pp. 47–56, 1963.
13. E. A. Spiegel, "A generalization of the mixing length theory of turbulent convection," Astrophys. J., vol. 138, no. 1, pp. 216–225, 1963.
14. A. T. Onufriev, "Use of the integral diffusion method for the investigation of turbulent transport," PMTF [Journal of Applied Mechanics and Technical Physics], no. 3, pp. 63–68, 1966.
15. A. S. Monin and A. M. Yaglom, Statistical Hydromechanics [in Russian], chap. I, Izd-vo Nauka, 1965.
16. G. Uhlenbeck and G. Ford, Lectures in Statistical Mechanics [Russian translation], Izd. Mir, 1965.
17. N. E. Kochin, I. A. Kibel, and N. V. Roze, Theoretical Hydromechanics [in Russian], chap. 2, Fizmatgiz, pp. 693–694, 1963.
18. P. L. Bhatnagar, E. P. Gross, and M. Krook, "A model for collision processes in gases," I. Phys. Rev., vol. 94, no. 3, pp. 511–525, 1954.
19. M. N. Kogan, "Equations of motion of a rarefied gas," PMM, vol. 22, no. 4, pp. 425–432, 1958.
20. H. W. Liepmann, R. Narasimha, and M. T. Chanine, "Structure of a plane shock layer," Phys. Fluids, vol. 5, no. 11, p. 1313–1324, 1962.
21. C. K. Chu, "Kinetic–theoretic description of the formation of a shock wave," Phys. Fluids, vol. 8, no. 1, pp. 12–22; no. 8, pp. 1450–1455, 1965.
22. A. N. Kolmogorov, "Equations of turbulent motion of an incompressible fluid," Izv. AN SSSR, Ser. fiz., vol. 6, no. 1–2, pp. 56–58, 1942.
23. V. G. Nevzglyadov, "The phenomenological theory of turbulence," Dokl. AN SSSR, vol. 47, no. 3, pp. 169–173, 1945.
24. L. Prandtl, "Über ein neues Formelsystem der ausgebildeten Turbulenz," Nachr. Gess. Wiss. Göttingen, Math.-Phys., pp. 6–19, 1945.
25. I. C. Rotta, "Statistiche Theorie nichtgomogener Turbulenz," Z. Physik, vol. 129, no. 5, pp. 547–572, vol. 131, no. 1, pp. 51–77, 1951.

25. I. C. Rotta, "Statistiche theorie nichtgomogener turbulenz," Z. Physik, vol. 129, no. 5, pp. 547–572, vol. 131, no. 1, pp. 51–77, 1951.

26. B. I. Davydov, "The statistical dynamics of an incompressible turbulent fluid," Dokl. AN SSSR, vol. 136, no. 1, pp. 47–50, 1961.
27. G. S. Glushko, "The turbulent boundary layer on a flat plate in an incompressible fluid," Izv. AN SSSR. Mekhanika, no. 4, pp. 13–23, 1965.
28. A. M. Obukhov, "Scale distribution of turbulence in flows of arbitrary section," PMM, vol. 6, no. 2–3, pp. 209–220, 1942.

29 September 1966 Novosibirsk